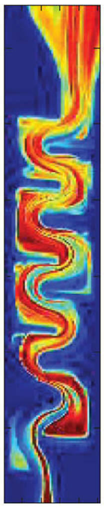
EPFL

Theoretical Microfluidics

MICRO-718

T. Lehnert and M.A.M. Gijs (EPFL-LMIS2)

EPFL - Lausanne
Doctoral Program in Microsystems and Microelectronics (EDMI)

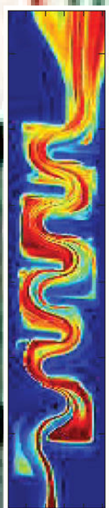


"Theoretical" microfluidics and more…

Course topics

Part I: *T. Lehnert* (EPFL-LMIS2)

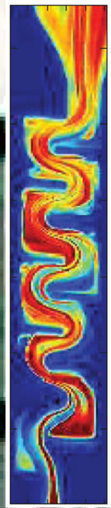
1. Introduction
2. Governing equations and flow solutions
3. Microfluidic channels and circuits
4. Diffusion and mixing in microscale
5. Capillary effects and microdroplets



Course topics

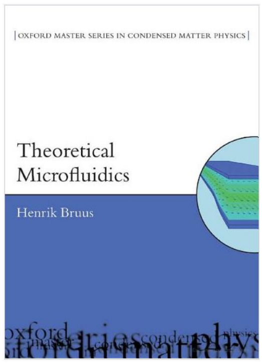
Part II: *Prof. M.A.M. Gijs* (EPFL-LMIS2)

1. Electrohydrodynamics
   Debye-layer, electro-osmotic flow, (di-)electrophoresis
2. Magnetophoresis
3. Nanofluidics



The theoretical parts of this course are mainly based on:

Theoretical Microfluidics
Henrik Bruus
Oxford University Press, 2008 (Reprint 2010)
(ISBN 978-0-19-923509-4)



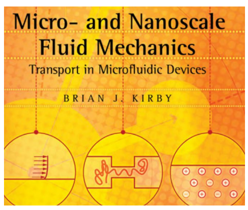
Equation/figure numbering in PART I of these lecture notes refers to the indicated edition of the book.

Numbering in PART II refers to an earlier edition !

All other references are indicated in the text.

Further reading:

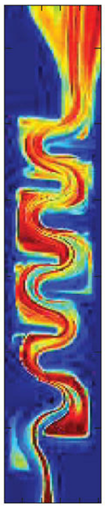
Micro- and nanoscale fluid mechanics: Transport in microfluidic devices
J. Kirby
Cambridge University Press 2010



also on youtube

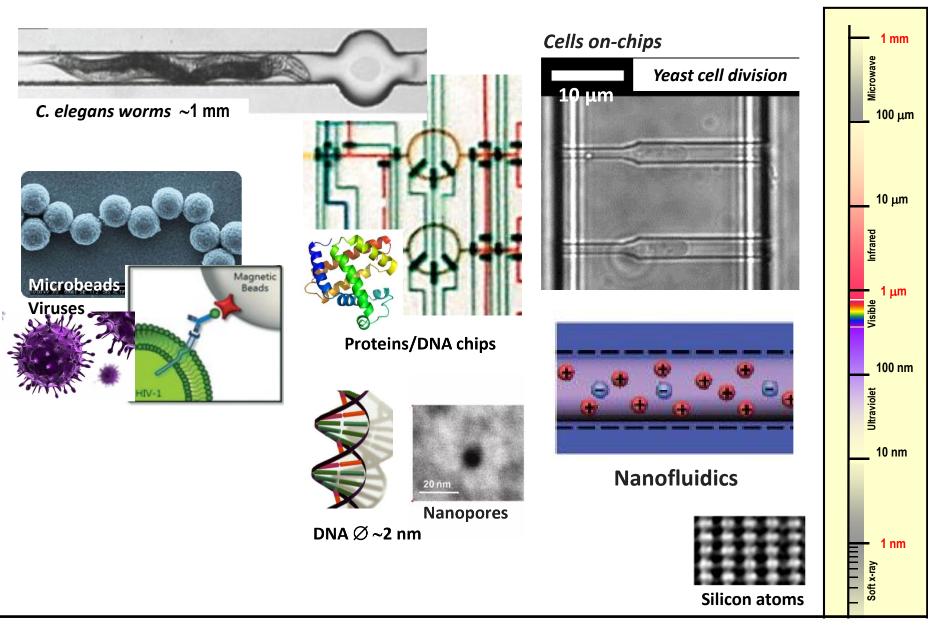
"Microfluidics" -- Thomas Lehnert -- EPFL (Lausanne)

1. INTRODUCTION



“Microfluidics” -- Thomas Lehnert -- EPFL (Lausanne)

Microfluidic and nanofluidic systems



Nature is based on powerful microfluidic systems

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Water transport system in vascular plants

Tallest known tree on earth: ***"****Hyperion"*

115.7 m ! Redwood (*sequoia sempervirens*)

The height is limited by increasing water transport constrains.

Microfluidics ?

$\Delta p(H_2O)$
10 bar

Transport of water occurs through **tube-like vessels with $\varnothing \approx$ 10 - 100 µm** (*Xylem*, ξυλον - wood).

Evaporation through **leaf pores ($\varnothing$ < 10 µm)** is a major driving force for pulling up through the tree trunk.

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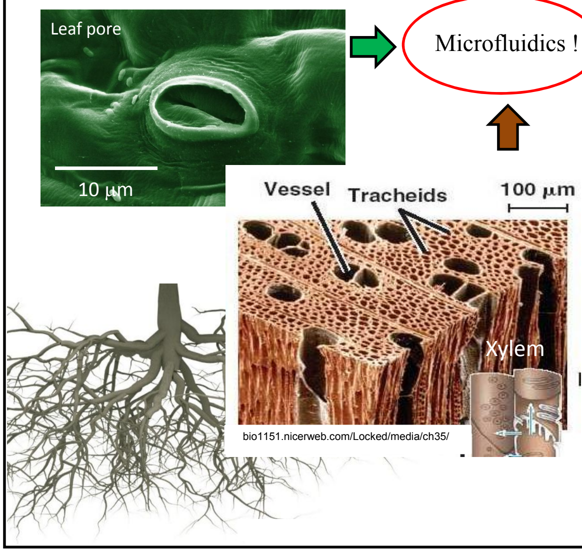


Microfluidics !

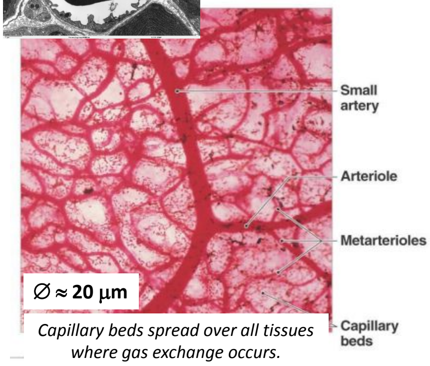
The human cardiovascular system

V ≈ 5L, flow rate at rest ≈ 5L/min

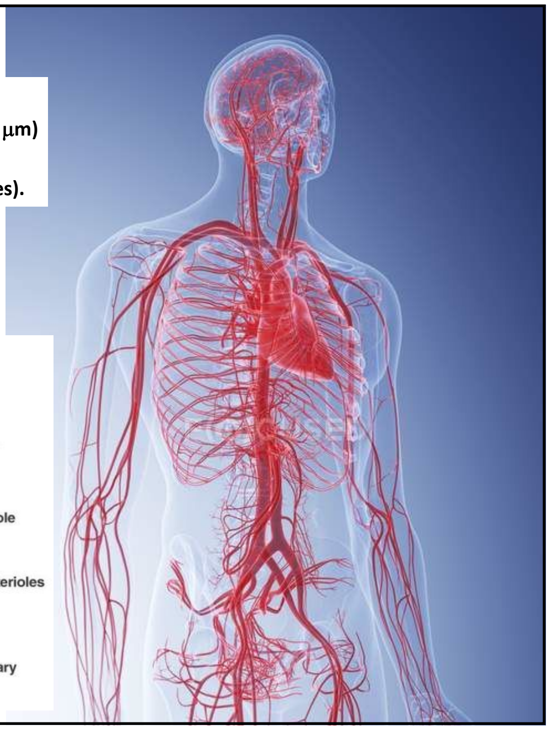
Networks of blood capillaries (∅ ≈ 5-10 µm) span over the lung and other organs.

Total length ~100000 km (!) (80% capillaries).

Red blood cell in a blood capillary



Capillary beds spread over all tissues where gas exchange occurs.



Macrofluidics - Microfluidics: What makes the difference ?

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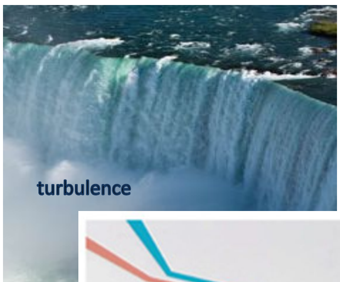
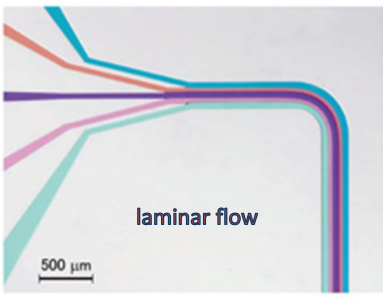
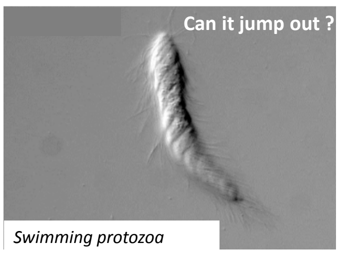
Flow patterns and solid body/liquid interactions

Niagara falls: Average flow rate 2000 m^3/s





Jumping humpback whale
L ≈ 15 m, speed 20-50 km/h. Large fins



Swimming protozoa

L ≈ 100 µm, speed ≤ 100 µm/s - Flagella, cilia

Macrofluidics - Microfluidics: What makes the difference ?

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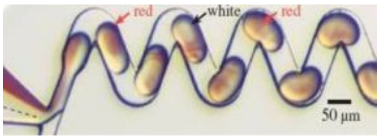
Size matters: Effects of downscaling are important !

$$\boxed{\frac{surface\ force}{volume\ force} \propto \frac{l^2}{l^3} = l^{-1} \underset{l\to 0}{\longrightarrow} \infty}$$

- ⇨ Viscous forces dominate (**surface forces**).
- ⇨ Inertial forces (and gravity) become negligible (**volume forces**).
- ⇨ **Interfacial/capillary forces** determine the liquid shape and driving forces.
- ⇨ **Exploiting boundary** (*e.g.* electrokinetic) **effects** is effective in microfluidic systems.
- ⇨ **Dimensionless numbers** evaluate the relative importance of competing forces.

Small raindrop
∅ ≈ mm, $V \approx 50$ µL



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2. Governing equations and flow solutions

2.1 Flow kinematics and shear stress

How does a fluid flow?

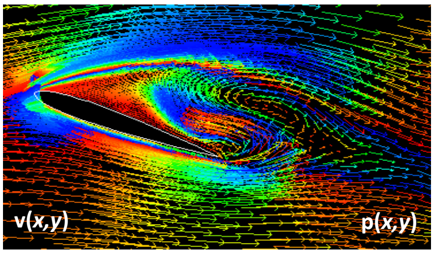
2.1 Flow kinematics and shear stress

Continuum description: *Flow around a wing*

v(**r**,*t*) field **Vector fields** (direction and length)

p(**r**,*t*) field **Scalar fields** ("value", no direction)

$\mathbf{v}(\mathbf{r},t) = \mathbf{v}(v_x, v_y, v_z, t) \equiv \mathbf{u}(\mathbf{r},t) = \mathbf{u}(u,v,w,t)$



- ⇨ A fluid deforms continuously at a **strain rate** ε, generated by velocity gradients $\nabla\mathbf{u}$.
- ⇨ **Extension** and/or **shear deformation**

Strain rate tensor ε $[s^{-1}]$

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

9 components in the general case for **u**(*u*,*v*,*w*)

- Example: Extensional strain rate of a fluidic element

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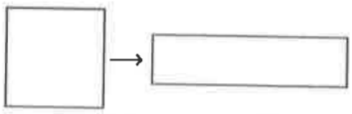
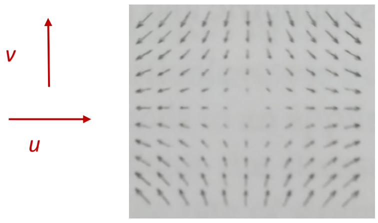
Strain tensor for pure extensional strain ?



$\mathbf{u}(u,v)$ $\boxed{u = x,\ v = -y}$



ε for a 2D velocity field $\mathbf{u}(u,v)$

$$\vec{\vec{\varepsilon}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_{ext} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Diagonal elements of $\boldsymbol{\varepsilon}$

$\Rightarrow$ **Extensional strain rate**

Σ = 0 for incompressible fluids !

here denotations are taken from: **J. Kirby, *Micro- and nanoscale fluid mechanics : transport in microfluidic devices***

- Example: Shear strain rate of a fluidic element

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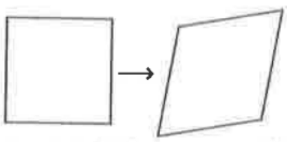
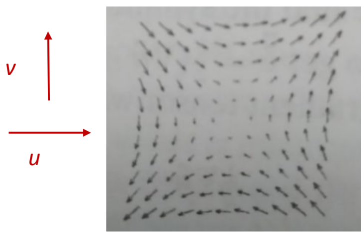
Strain tensor for pure shear strain ?



$\mathbf{u}(u,v)$ $\boxed{u = y,\ v = x}$



ε for a 2D velocity field $\mathbf{u}(u,v)$

$$\vec{\vec{\varepsilon}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

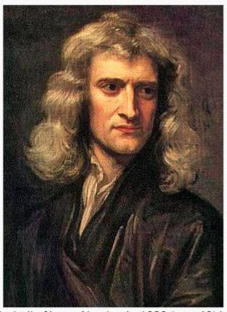
$$\boldsymbol{\varepsilon}_{shear} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Symmetric off-diagonals

$\Rightarrow$ **Shear strain rate**

here denotations are taken from: **J. Kirby, *Micro- and nanoscale fluid mechanics : transport in microfluidic devices***

Sir Isaac Newton



Portrait of Isaac Newton in 1689 (age 46) by Godfrey Kneller

Born	25 December 1642 [NS: 4 January 1643][1] Woolsthorpe, Lincolnshire, England
Fields	Physics · Natural philosophy Mathematics · Astronomy Alchemy · Christian theology Economics

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⇨ **The strain rate tensor ε is related to the stress tensor σ** [N/m² = Pa] (stress ≡ force / surface area)

$$\boxed{\sigma = \eta\, \varepsilon}$$

NEWTON'S LAW OF VISCOSITY

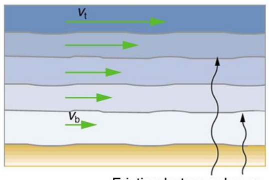
$\sigma \equiv \tau$ *in literature !*

η (T) or μ (T) [Pa·s] is the dynamic viscosity

η = constant (for T = const) ⇨ **Newtonian fluids**

In microfluidics the viscous flow regime (internal friction) is predominant.

⇨ σ is of fundamental importance!



Friction between layers

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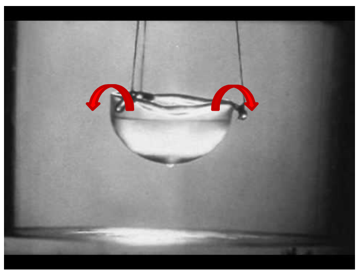
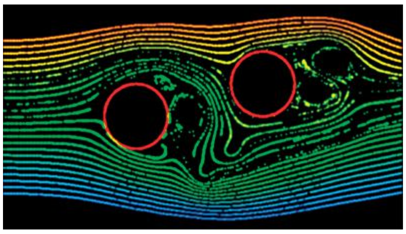
η = 0



^{4}He (superfluid) at T < 2.17 K ≈ -271 °C

2.1 Flow kinematics and shear stress



- ⇨ Partial differential equations describing local properties of the flow field.
- ⇨ in particular for **v**(**r**,*t*) *p*(**r**,*t*) and force densities.

Governing equations

- ⇨ Continuity equation (mass conservation)
- ⇨ Navier-Stokes equations for v(*r*,*t*) (momentum conservation)
- ⇨ Convection-Diffusion equation

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2.2 The continuity equation in fluid dynamics

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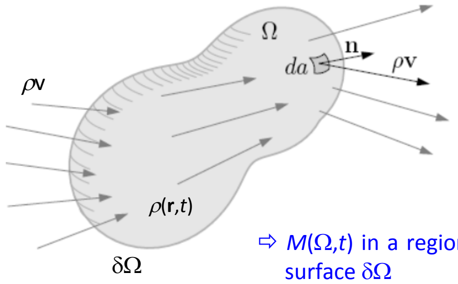
2.2 Continuity equation in fluid dynamics

- The continuity eqn expresses the conservation of mass $M(\Omega,t)$



$$M(\Omega,t) = \int_{\Omega} \mathrm{d}\mathbf{r}\;\rho(\mathbf{r},t) \qquad (2.2)$$

$$\mathbf{J}(\mathbf{r},t) = \rho(\mathbf{r},t)\;\mathbf{v}(\mathbf{r},t) \qquad (2.3)$$

Mass flux density $\mathbf{J}(\mathbf{r},t)$ [kg/(m^{2}s)]

mass density ρ, flow velocity $\mathbf{v}$

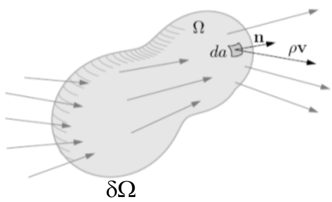
⇨ $M(\Omega,t)$ in a region Ω can only vary by mass flow through the surface $\delta\Omega$

$$\partial_t M(\Omega,t) = \int_{\Omega} \mathrm{d}\mathbf{r}\;\partial_t\rho(\mathbf{r},t) = -\int_{\partial\Omega} \mathrm{d}a\;\mathbf{n}\cdot\;\mathbf{J}(\mathbf{r},t) \qquad (2.4)$$

$$\int_{\Omega} \mathrm{d}\mathbf{r}\;\partial_t\rho(\mathbf{r},t) = -\int_{\partial\Omega} \mathrm{d}a\;\mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big)$$

2.2 Continuity equation in fluid dynamics

$$\int_{\Omega} \mathrm{d}\mathbf{r}\;\partial_t\rho(\mathbf{r},t) = -\int_{\partial\Omega} \mathrm{d}a\;\mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big)$$

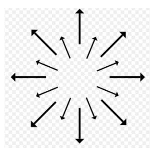


Solve this for v(r,t): What is the problem ?

$$-\int_{\partial\Omega} \mathrm{d}a\;\mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) = -\int_{\Omega} \mathrm{d}\mathbf{r}\;\boldsymbol{\nabla}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) \qquad (2.5)$$

⇨ using the **Gauss theorem**

$$\boxed{\int_{\partial\Omega} \mathrm{d}a\;\mathbf{n}\cdot\mathbf{V} = \int_{\Omega} \mathrm{d}\mathbf{r}\;\boldsymbol{\nabla}\cdot\mathbf{V}}$$



A vector field with divergence has a source (or a sink)

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2.2 Continuity equation in fluid dynamics

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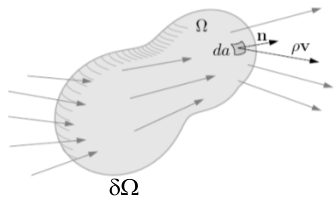
$$\int_{\Omega} \mathrm{d}\mathbf{r}\, \partial_t \rho(\mathbf{r},t) = -\int_{\partial\Omega} \mathrm{d}a\, \mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big)$$



Solve this for v(r,t): What is the problem ?

$$-\int_{\partial\Omega} \mathrm{d}a\, \mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) = -\int_{\Omega} \mathrm{d}\mathbf{r}\, \boldsymbol{\nabla}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) \qquad (2.5)$$

- **Nabla operator** $\boldsymbol{\nabla} \equiv \mathrm{e}_x\partial_x + \mathrm{e}_y\partial_y + \mathrm{e}_z\partial_z$
- **The divergence of a vector field is a scalar field** $\nabla\cdot\mathbf{v} \equiv \partial_x v_x + \partial_y v_y + \partial_z v_z$
- **The gradient of a scalar field is a vector field** $\nabla p = \mathrm{e}_x\partial_x p + \mathrm{e}_y\partial_y p + \mathrm{e}_z\partial_z p$

2.2 Continuity equation in fluid dynamics

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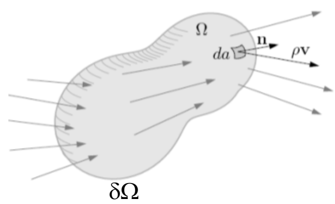
$$\int_{\Omega} \mathrm{d}\mathbf{r}\, \partial_t \rho(\mathbf{r},t) = -\int_{\partial\Omega} \mathrm{d}a\, \mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big)$$



Solve this for v(r,t): What is the problem ?

$$-\int_{\partial\Omega} \mathrm{d}a\, \mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) = -\int_{\Omega} \mathrm{d}\mathbf{r}\, \boldsymbol{\nabla}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) \qquad (2.5)$$

with (2.4) = (2.5)

$$\boxed{\int_{\Omega} \mathrm{d}\mathbf{r}\, \Big[\partial_t \rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big)\Big] = 0} \qquad (2.6)$$

Integral form of the Continuity equation

2.2 Continuity equation in fluid dynamics

It describes the mass balance in any point of the 3D flow field.

- for compressible fluids with $\rho(\mathbf{r},t)$ and a flow field $\mathbf{v}(\mathbf{r},t)$

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \mathrm{v}) = 0 \quad \text{or} \quad \partial_t \rho + \nabla \cdot \mathbf{J} = 0} \qquad (2.7)$$

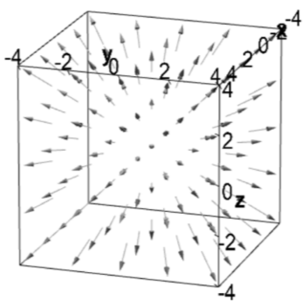
- for incompressible fluids
  (set ρ = const and uniform, *i.e.* $\partial_t \rho = 0$ and $\partial_i \rho = 0$)

Divergence of $\mathbf{v}(\mathbf{r},t)$

$$\boxed{\nabla \cdot \mathrm{v} = 0 \quad \text{or} \quad \mathrm{div}\ \mathbf{v} = 0} \qquad (2.9)$$

2.2 Continuity equation in fluid dynamics

Examples: A physical flow field must fulfil the continuity equation ($\nabla \mathbf{v} = 0$). For incompressible fluids the divergence of $\mathbf{v}(\mathbf{r},t)$ is zero everywhere in the field (no source, no sink).



$$\mathbf{v}(x,y,z) = (x,y,z)$$

$$\mathrm{div}\ \mathbf{v}(x,y,z) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1+1+1 = 3$$

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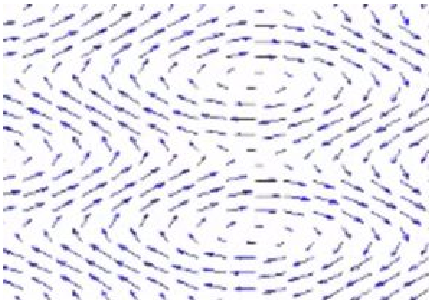
2.2 Continuity equation in fluid dynamics



$$\mathbf{v}(x,y,z) = \begin{pmatrix} \sin k_x y \\ \cos k_y x \\ 0 \end{pmatrix}$$

$$\nabla \equiv \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y + \mathbf{e}_z \partial_z$$

div **v** = 0

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2.3 Navier-Stokes equations

Claude-Louis Navier

Bust of Claude Louis Marie Henri Navier at the École Nationale des Ponts et Chaussées

Born	10 February 1785 Dijon, France
Died	21 August 1836 (aged 51) Paris, France
Nationality	French

A specialist in bridge building (he was the first to develop a theory of suspension bridges).

1821/1822

Navier modified the *Euler equations* (1757)for inviscid flow

$$\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \qquad + \rho\,\mathbf{g}$$

…by introducing friction in the equations of fluid motion.

$$+\eta\nabla^2\mathbf{v}$$

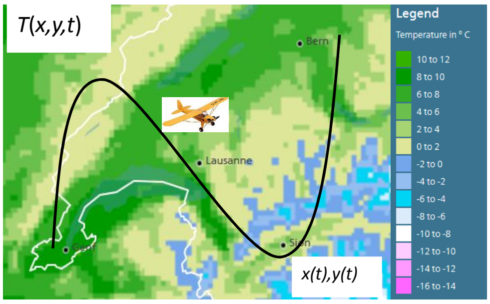
viscous term

- The Navier-Stokes equations (NSE) describe the fluidic transport by advection
  ⇨ Equations of motion for a flow field **v**(**r**,*t*).
- They express conservation of momentum.
  ⇨ Newton's 2nd law applied to fluid mechanics.

Navier-Stokes eqns derived by using the Lagrange derivative

Consider a particle moving on an arbitrary path though a 2-D field $\Phi(x,y,t)$, *e.g.* a *T* or *p* field. Φ depends on (x,y) but may also change with time *t.*

Lagrangian description: The observer moves on the particle through the field.



How do the field parameters Φ (x,y,t) change along the path ?

Navier-Stokes eqns derived by using the Lagrange derivative

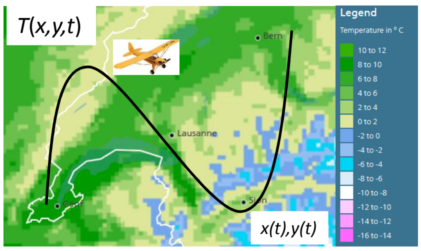
The variation of $\Phi(x(t),y(t),t)$ along the pathline is expressed by the total time derivative of Φ.

⇨ **Lagrange derivative DΦ/Dt**

(also called substantial or material derivative)

⇨ The pathline is given by [*x(t),y(t))*]

⇨ Applying the chain rule for deriving $\Phi(r(t),t)$



(2D⇨3D)

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} \equiv \frac{\partial\Phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\Phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\Phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial\Phi}{\partial t}$$

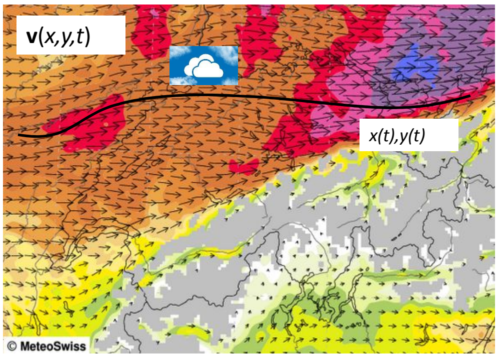
or with $v_i = dx_i/dt$

$$\boxed{\frac{\mathrm{D}\Phi}{\mathrm{D}t} \equiv \frac{\partial\Phi}{\partial x}v_x + \frac{\partial\Phi}{\partial y}v_y + \frac{\partial\Phi}{\partial z}v_z + \frac{\partial\Phi}{\partial t}}$$

The 3D **Lagrange derivative** can be written as $\boxed{D_t = \partial_t + (\mathbf{v}\cdot\boldsymbol{\nabla})}$ (2.34)

Navier-Stokes Eqns derived by using the Lagrange derivative

How does the velocity of a particle change on a specific path through a flow field $\mathbf{v}(\mathbf{r},t)$ *?*



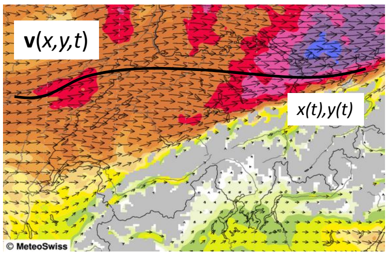
In this case trajectory and velocity of the particle are not arbitrary but determined by the flow field $\mathbf{v}(\mathbf{r},t)$ itself !

Navier-Stokes Eqns derived by using the Lagrange derivative

The Lagrange derivative $D_t = \partial_t + (\mathbf{v}\cdot\boldsymbol{\nabla})$ for the velocity component $v_x(x,y,z,t)$ of the particle is given by (likewise for v_y and v_z):

$$\frac{\mathrm{D}v_x}{\mathrm{D}t} \equiv \frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y + \frac{\partial v_x}{\partial z}v_z + \frac{\partial v_x}{\partial t}$$

⇨ Describes the acceleration of the particle when moving through the flow field $\mathbf{v}(\mathbf{r},t)$.



Newton's 2nd law for a free particle $m\,\mathrm{d}_t\mathbf{v} = \sum_j \mathbf{F}_j$

⇨ Relating this to forces

"Newton's 2nd law" for a fluidic parcel $\rho\, D_t\mathbf{v} = \sum_j \mathbf{f}_j$

⇨ The equation of motion of the fluidic parcel takes the form of the Navier-Stokes equation

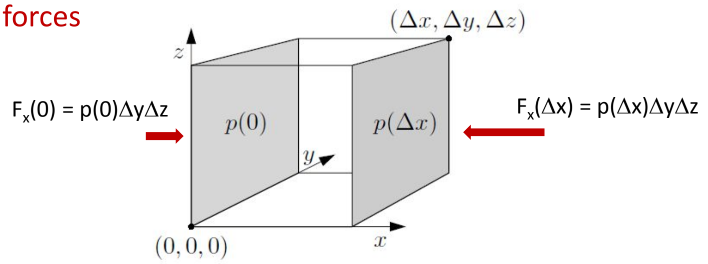
$$\boxed{\rho\big[\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big] = \sum_j \mathbf{f}_j,} \qquad (2.35)$$

⇨ The force densities $\mathbf{f}_j$ are related to pressure, viscosity and external body forces.

A heuristic derivation of the pressure and viscosity force densities

EPFL

Pressure forces



Total pressure force in *x* $\quad F_x = p(0)\,\Delta y\Delta z - p(\Delta x)\,\Delta y\Delta z$

Total force density in *x*

$$f_x = \frac{p(0)\,\Delta y\Delta z - p(\Delta x)\,\Delta y\Delta z}{\Delta x\Delta y\Delta z} = -\frac{p(\Delta x) - p(0)}{\Delta x} \underset{\Delta x\to 0}{\longrightarrow} -\partial_x p, \qquad (2.74)$$

which is the x component of $\mathbf{f} = -\boldsymbol{\nabla} p$. The other two components are derived similarly.

$$\boxed{\mathbf{f}(x,y,z) = -\nabla p(x,y,z)}$$

A heuristic derivation of the pressure and viscosity force densities

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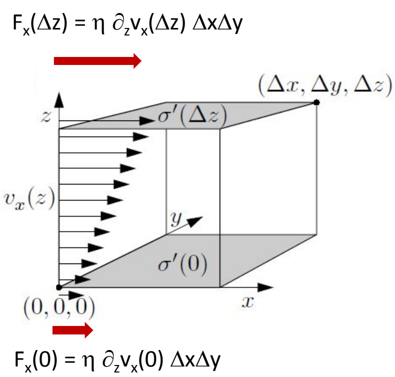
Viscous shear forces



Viscous force density

$$\boxed{\mathbf{f}(x,y,z) = \eta\nabla^2\mathbf{v}(x,y,z)}$$

$$f_x = \eta\frac{\partial_z v_x(\Delta z)\,\Delta x\Delta y - \partial_z v_x(0)\,\Delta x\Delta y}{\Delta x\Delta y\Delta z} = \eta\frac{\partial_z v_x(\Delta z) - \partial_z v_x(0)}{\Delta z} \underset{\Delta z\to 0}{\longrightarrow} \eta\partial_z^2 v_x, \quad (2.75)$$

EPFL

Navier-Stokes equations for incompressible fluids

ρ = const, η = const

$$\boxed{\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v} + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E}} \qquad (2.30)$$

Non-linear 2nd order vector partial differential eqn for $\mathbf{v}(\mathbf{r},t)$

Navier-Stokes equations for compressible fluids

More details in *Henrik Bruus* "Theoretical Microfluidics"

$\rho(\mathbf{r},t)$ and η = const, ζ = const

$$\boxed{\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v} + \Big(\tfrac{1}{3}\eta + \zeta\Big)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E}} \qquad (2.29)$$

η is the dynamic viscosity due to shear stress.

ζ stands for internal friction due to compression.

B

EPFL

Navier-Stokes equations for incompressible fluids (vector form)

Advective terms account for **acceleration** of fluidic particles in unsteady or steady flow states.
⇨ **Inertial force densities**

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v} + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E}$$

The transient term $\partial_t\mathbf{v}$ is relevant if $\mathbf{v}(t)$ changes with time.

The non-linear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ describes convective acceleration (time-independent), *e.g.* in systems with no translation invariance.

$(\mathbf{v}\cdot\nabla)\mathbf{v}$ is particularly relevant in turbulent flow regimes.

B

"Microfluidics" -- Thomas Lehnert -- EPFL (Lausanne)

Navier-Stokes equations for incompressible fluids (vector form)

Body force densities

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v} + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E}$$

∇p and $\nabla\cdot\sigma$ are surface force densities for pressure and viscous shear stress.

For incompressible fluids $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \eta\boldsymbol{\nabla}^2\mathbf{v}$

η dynamic viscosity [Pa·s]

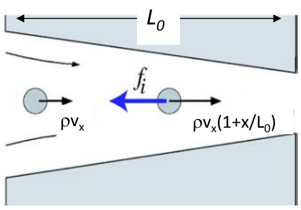
Exploring the non-linear term $(v\cdot\nabla)v$ in the Navier-Stokes eqns

Example: Steady (time independent) flow through a constriction (nozzle)

⇨ Advective acceleration of the flow

⇨ The inertial part of the NSE is given by

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) \quad \textit{with} \quad \partial_t \mathbf{v} = 0$$



T. M. Squires and S. R. Quake: Microfluidics: Fluid physics at the nanoliter scale

Rough estimate of the inertial force density f_i, assuming that v_x increases by V_0 over L_0.

V_0 and L_0 are characteristic scales of the system.

$$f_i \sim \frac{\rho V_0^2}{L_0}$$

[kg/m³ · m²/s²m]
[kg m/s² · m⁻³]
[N · m⁻³]

Exploring the non-linear term (v·∇)v in the Navier-Stokes eqns

EPFL

Example: Inviscid flow (neglecting viscosity) in a pressure field.

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p$$

e.g. only pressure gradient in *x*-direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x}$$

A pressure force in *x*-direction generates velocity/gradient components in *x,y,z* directions. This results in flow instabilities and turbulences !

Turbulences on marcoscale due to fluidic inertia

In microfluidics inertial forces are normally negligible with respect to viscous forces.
Some examples where inertial fluidic properties are relevant will be shown later.

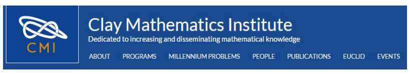
B 3/00

2.4 Solutions for simple flow problems

EPFL

US$ 1 million *Millennium Problem*

Navier-Stokes equations are a system of non-linear coupled partial differential eqns.



http://www.claymath.org/millennium-problems

Approaches for solving the Navier-Stokes equations

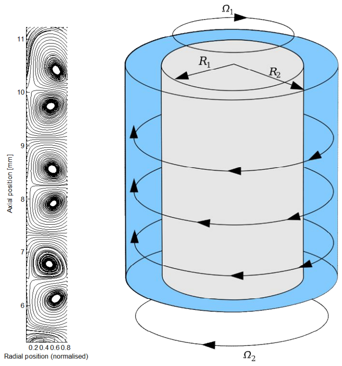
- Numerical solutions
- Analytical techniques (*e.g.* eigenfunction expansion)
- Simplifications in specific cases (simple geometries, Stokes flow at low flow rates)

⇨ Initial and boundary conditions have to be defined.

$$\boxed{\mathbf{v}(\mathbf{r}) = \mathbf{0}, \quad \text{for } \mathbf{r} \in \partial\Omega \;\; \text{(no-slip)}}$$

No-slip boundary condition for a channel wall

Taylor-Couette flow of a viscous fluid between rotating cylinders



Good model system to study flow instabilities and transitions.

Low rotation speed

Couette flow purely azimuthal and laminar (bearing flow).

Application: Rheometers for measuring η

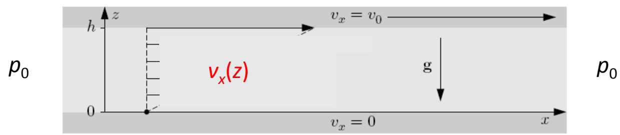
High rotation speed

Flow becomes unstable: Vortices and turbulent patterns emerge.

https://doi.org/10.1051/epjconf/201921302014

https://en.wikipedia.org/wiki/Taylor-Couette_flow

Couette flow between two moving parallel plates



(Fig. 3.3)

Navier-Stokes eqn

$$\boxed{\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{v} + \rho\, \mathbf{g} + \rho_{\mathrm{el}} \mathbf{E}}$$

(i) Translation invariance along *x* $\quad \mathbf{v}(\mathbf{r}) = v_x(z)\, \mathbf{e}_x$

⇨ *The non-linear term vanishes* $\quad (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = v_x(z)\partial_x\big[v_x(z)\big] = 0$

(ii) $p = p_0$ = const

(iii) Steady state conditions $\quad \partial_t \mathbf{v} = 0$

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Couette flow between two moving parallel plates

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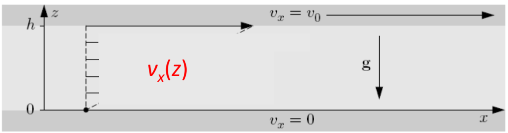
p_0



p_0

(Fig. 3.3)

Navier-Stokes eqn

$$\boxed{\eta\, \partial_z^2 \mathbf{v} = 0}$$

(i) Translation invariance along *x* $\quad \mathbf{v}(\mathbf{r}) = v_x(z)\, \mathbf{e}_x$

⇨ *The non-linear term vanishes* $\quad (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = v_x(z)\partial_x\big[v_x(z)\big] = 0$

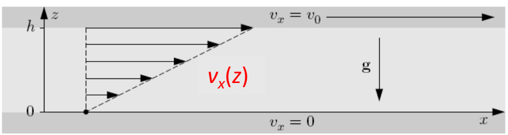
(ii) $p = p_0$ = uniform

(iii) Steady state conditions $\quad \partial_t \mathbf{v} = 0$

(iv) no external forces

Couette flow between two moving parallel plates

EPFL



(Fig. 3.3)

Navier-Stokes eqn

$$\boxed{\eta\, \partial_z^2 \mathbf{v} = 0} \quad (3.13)$$

Solution by integration

⇨ Linear velocity profile

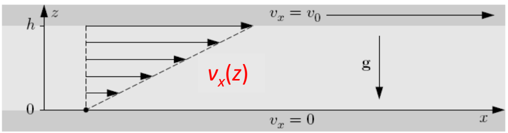
$$v_x(z) = v_0\, \frac{z}{h} \quad (3.15)$$

Force required to move the plate (with surface *A*)

$\sigma_{xz} = \eta\, \partial v_x / \partial z$

$$F_x = \sigma'_{xz}\, \mathcal{A} = \eta\, \frac{v_0 \mathcal{A}}{h} \quad (3.16)$$

Couette flow between two moving parallel plates



(Fig. 3.3)

$$\boxed{\eta\, \partial_z^2 \mathbf{v} = 0}$$

$$v_x(z) = v_0\, \frac{z}{h}$$

$$F_x = \sigma'_{xz}\, \mathcal{A} = \eta\, \frac{v_0 \mathcal{A}}{h}$$

Physical interpretation of Couette flow:

(i) No convective acceleration, uniform pressure.

(ii) The viscous force density is zero (the second derivative of **v**).

(iii) The profile **v(r)** is independent of η.

(iv) The total force F_x is fct of the viscosity.

2.5 Reynolds number and Stokes flow

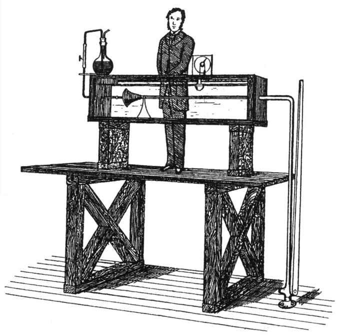
Osborne Reynolds

Born	23 August 1842 Belfast, Ireland
Died	21 February 1912 (aged 69) Watchet, Somerset, England
Nationality	United Kingdom
Fields	Physics

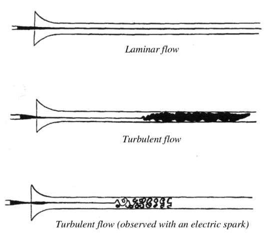
Philosophical Transactions of the Royal Scociety 1883

84 Mr. O. Reynolds. [Mar. 15,

III. " An Experimental Investigation of the Circumstances which Determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels." By OSBORNE REYNOLDS, F.R.S. Received March 7,



Flow regimes



Reynolds number

$$Re = \frac{F_i}{F_v} = \frac{\rho l v}{\mu}$$

for $Re \geq Re_{crit} >> 1 \Rightarrow$ turbulent flow regime

Different flow regimes

EPFL

Transition from laminar to turbulent (water) flow in a tube with increasing flow speed

(Reproduction of Reynold's original experiment)

https://www.youtube.com/watch?v=XOLI2KeDiOg

Dimensionless form of the Navier-Stokes equations

EPFL

Navier-Stokes eqn $$\boxed{\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v}}$$

The general fluidic properties of a system can be evaluated by using **characteristic scales** determined by the boundary conditions: $\boldsymbol{L_0}$ **,** $\boldsymbol{V_0}$

Dimensionless (normalized) forms can be derived for all variables, in particular for …

$$\mathbf{r} = L_0\,\tilde{\mathbf{r}} \qquad \boldsymbol{\nabla} = (1/L_0)\,\tilde{\boldsymbol{\nabla}}$$
$$\mathbf{v} = V_0\,\tilde{\mathbf{v}}$$

Pressure p is also a "stress"

In microfluidics, the pressure p is normalized by a characteristic shear stress $\eta V_0/L_0$.

$$p = \frac{\eta V_0}{L_0}\,\tilde{p} = P_0\,\tilde{p} \qquad (2.36)$$

Dimensionless form of the Navier-Stokes equations

EPFL

Navier-Stokes eqn
for $\partial_t \mathbf{v} = 0$

$$\rho\Big((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla}p + \eta\nabla^2\mathbf{v}$$

⇨ Making the Navier-Stokes eqns dimensionless

$$\rho\left[\frac{V_0^{\,2}}{L_0}\,(\tilde{\mathbf{v}}\cdot\tilde{\boldsymbol{\nabla}})\tilde{\mathbf{v}}\right] = -\frac{P_0}{L_0}\,\tilde{\boldsymbol{\nabla}}\tilde{p} + \frac{\eta V_0}{L_0^{\,2}}\,\tilde{\boldsymbol{\nabla}}^2\tilde{\mathbf{v}} \qquad (2.37)$$

Dimensionless form of the NSE

$$\boxed{Re\Big[(\tilde{\mathbf{v}}\cdot\tilde{\boldsymbol{\nabla}})\tilde{\mathbf{v}}\Big] = -\tilde{\boldsymbol{\nabla}}\tilde{p} + \tilde{\boldsymbol{\nabla}}^2\tilde{\mathbf{v}}} \qquad (2.38)$$

Reynolds number

$$\boxed{Re = \frac{\rho\, V_0\, L_0}{\eta}} \;\Rightarrow\; \frac{\rho V_0^{\,2}/L_0}{\eta V_0/L_0^{\,2}} \;\rightarrow\; \frac{inertial\ force}{viscous\ force} \qquad .39)$$

or the kinematic viscosity $\nu = \eta/\rho$, $\nu = 10^{-6}$ m²/s for water

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Some values for Reynolds numbers

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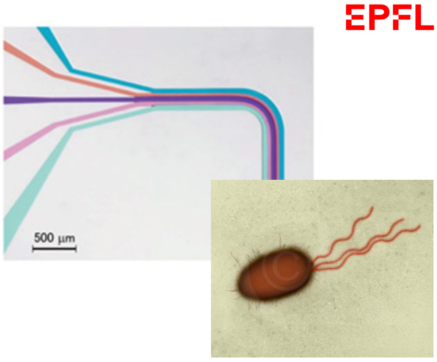


Typical values for microfluidic devices

Water-based fluids $\eta \approx 1.0$ mPa s
Range of flow speeds 1 µm/s - 1 cm/s
Typical channel width 1 - 100 µm

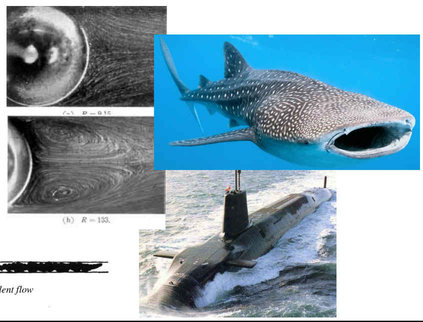
⇨ *Re* range between $\mathcal{O}(10^{-6})$ to $\mathcal{O}(10^{0})$
⇨ *Re* are very small in microfluidic systems

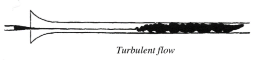
Microorganisms ~ 10^{-6} - 10^{-3}

Blood flow in aorta ~ 1×10^3
Human swimming ~ 10^4 - 10^6

Onset of turbulent flow

Flow in a pipe 2.3×10^3 to 5.0×10^4
Boundary layers up to 10^6





Inertial microfluidics

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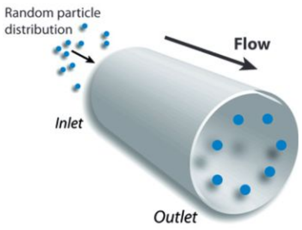
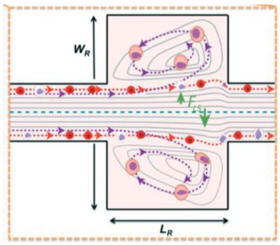
- ⇨ In microfluidics fluid inertia is normally negligible (Stokes flow, $Re \ll 1$).
- ⇨ Inertial microfluidics works in between Stokes and turbulent regimes (inertia and fluid viscosity are finite, $\sim 1 < Re < \sim 100$).



Particle sorting by inertial lift forces



Vortices in expanding channels for cell trapping

J. Zhang et al., Fundamentals and applications of inertial microfluidics: a review, Lab Chip, 2016, 16, 10

EPFL

3D hydrodynamic focusing using Dean flow

Dean flow: Circulating secondary flow in slightly curved channels ($R >> w$).

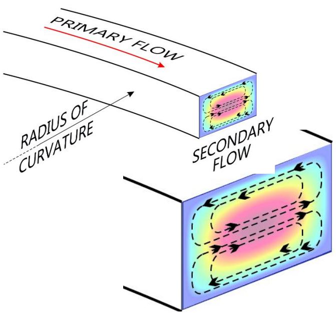
Inertial effects at high Re numbers ($\sim 1 < Re < \sim 100$) / high flow speeds !

Inhomogeneous flow profile causes centrifugal forces (greatest in the center).

Dean number

$$\kappa = (d/R)^{0.5}\, Re$$

tube diameter d or channel width, R is the radius of curvature of the path of the channel.



Dean flow in curved channels

(see also Chapter 4.2.5: "*A multivortex mixer based on inertial flow properties*")

3D hydrodynamic focusing using Dean flow

Example of a single-layer planar device (PDMS)

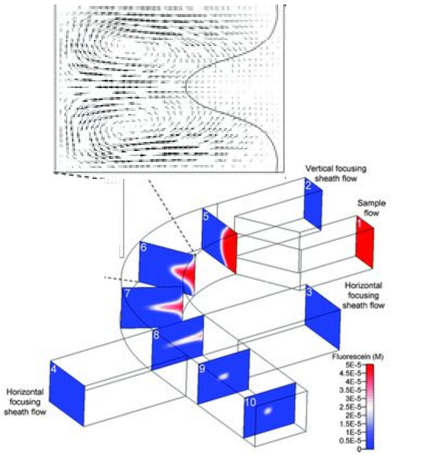
⇨ Dean vortices generate "microfluidic drifting"

⇨ Stretching of the sample flow across the channel width (vertical focusing, red).

⇨ Two lateral sheath flows are introduced for horizontal focusing.

Re = 74 (!), *De* ≈ 43

High flow speed in the range of ≈ m/s !



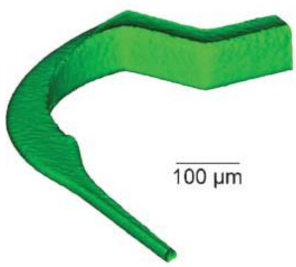
Cross-sectional profiles of the fluorescein dye concentration in the focusing device. Inset: simulation of the secondary flow velocity field shows Dean vortices in the 90° curve.

Main channel w=100 μm, h=75 μm, L=1cm, R_{curve} = 250 μm

X. Mao et al., Lab Chip, 2007, 7, 1260-1262

X. Mao et al., Lab Chip, 2009, 9, 1583-1589

3D hydrodynamic focusing using Dean flow

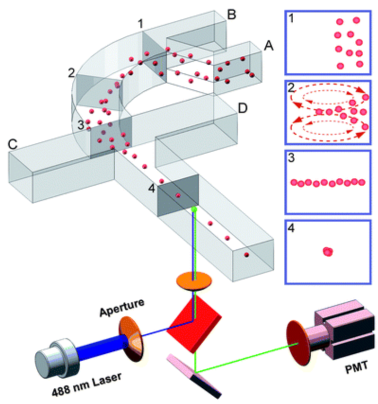


Focused beam ∅ ≤ 15 μm

The 3D architecture of the sample flow during the focusing process characterized by confocal microscopy (fluorescein solution).

X. Mao et al., Lab Chip, 2007, 7, 1260-1262

3D focusing for on-chip cytometry



Inlet A: Cells or particles;

Inlet B: vertical focusing sheath flow;

Inlets C and D: horizontal focusing sheath flows.

Inset 2 represent the Dean vortices.

Stokes flow and Stokes equations

EPFL

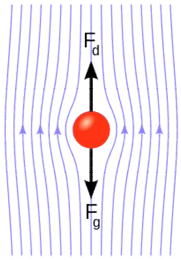
Sir George Stokes, Bt.

Born	13 August 1819 Skreen, County Sligo, Ireland
Died	1 February 1903 (aged 83) Cambridge, England
Fields	Mathematics and physics
Institutions	University of Cambridge

Stokes, G. G.
Effect of the Internal Friction of Fluids on the Motion of Pendulums
Transactions of the Cambridge Philosophical Society, Vol. 9, pp. 8-93, 1851

He assumed that the flow is so slow that acceleration of the fluid as it passes around the sphere can be ignored, $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$.

He derived the **viscous drag force $\mathbf{F_d}$ on a sphere** by solving a simplified version of the Navier-Stokes eqns analytically.



Sedimentation of (micro-)particles

Stokes flow and Stokes equations

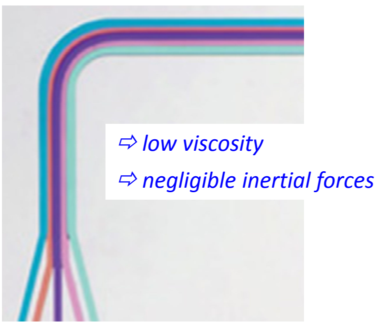
EPFL

Stokes equation

$$\boxed{0 = -\nabla p + \eta \nabla^2 \mathbf{v}} \qquad (2.41)$$

Stokes flow or "creeping flow" (very slow !)

Stokes equations are most relevant for *flow in microchannels,* transport of microparticles or cells, the swimming of microorganisms, etc.



High viscosity flows or very slow motions.

Stokes flow and Stokes equations

Stokes equation

$$\boxed{0 = -\nabla p + \eta \nabla^2 \mathbf{v}} \quad (2.41)$$

Stokes flow or "creeping flow" (very slow !)

Reynolds introduced "his" number only in 1883, i.e. more than 30 after Stokes' intuitive approach.

For *Re* << 1 the non-linear term $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the Navier-Stokes equation can be neglected.

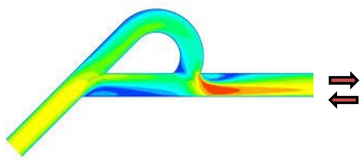
$$Re\left[(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right] = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{\mathbf{v}}$$

Re < 0.1 is a rule of thumb that the Stokes eqns are a good approximation.

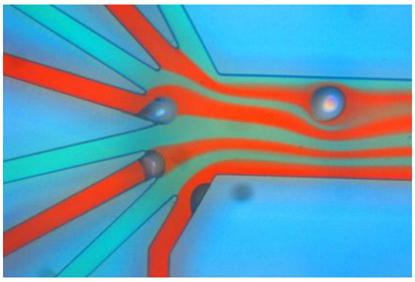
"Microfluidics" -- Thomas Lehnert -- EPFL (Lausanne)

Properties of Stokes flow

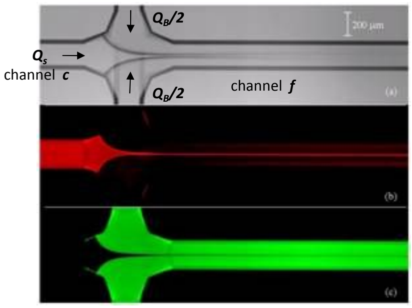
- Uniqueness (no flow instabilities)
  ⇨ **Laminar flow patterns**
- Linear in *p* and **v** : Superposability of flow solutions, *e.g.* for a changing driving force (boundary condition).
- Reversibility: (i) Flow symmetry around obstacles. (ii) If a boundary motion is reversed then each point of the flow retraces its history.



Tesla valve. In the Stokes flow regime no "valving" effect is observed for inverted flow directions as forward and reverse flow paths



Microfluidic artwork showing laminar flow patterns.



Very stable flow conditions:
Hydrodynamic focusing down to $w_{fs} \approx 10\ \mu m$

The transient form of the Stokes equation

EPFL

$\partial_t \mathbf{v} \neq 0$ ⇨ Transient form of the Stokes equation

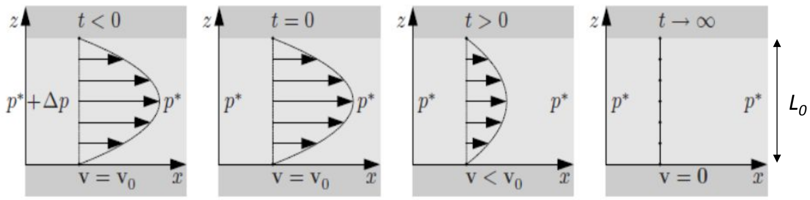
$$\boxed{\rho \partial_t \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v}}$$

Example: Poiseuille flow in a channel for time dependent boundary conditions: **Δp = 0 for $t \geq 0$ (relaxing flow)**

Stokes eqn takes the form of a momentum diffusion equation with the diffusion constant

$\nu = \eta/\rho$ (kinematic viscosity [m²/s])

$$\boxed{\partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v}}$$



The transient form of the Stokes equation

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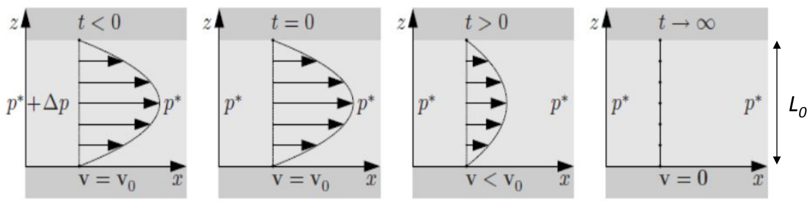
Estimation of the **time scale τ_0 to establish/or to stop** a steady laminar flow upon application/release of an external pressure difference Δp.



$$\boxed{\partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v}}$$

Balance of unsteady inertial and viscous force densities

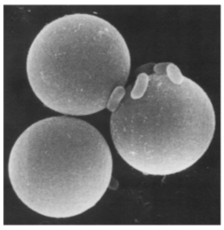
$$f_i \sim \frac{\rho V_0}{\tau_0} \qquad f_v \sim \frac{\eta V_0}{L_0^2} \qquad \Longrightarrow \tau_0 \sim \frac{\rho L_0^2}{\eta}$$

≈ 10 ms for a 100 μm channel

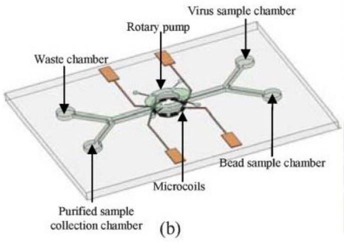
Unbounded Stokes flow around a sphere: The viscous drag force

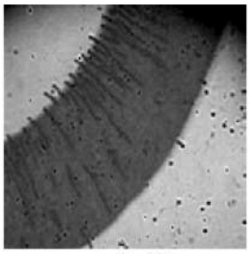
Important for transport of microparticles or cells, diffusivity of macromolecules, the swimming of microorganisms, etc.

Example: Microsystems based on functionalized magnetic microbeads



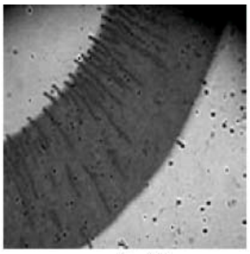
Y. enterocolitica attached to magnetic microbeads (∅ 4.5 μm)





K. Lien et al., Lab Chip, 2007, 7, 868–875 | 869
DOI: 10.1039/b700516d

Unbounded Stokes flow around a sphere: The viscous drag force



Purification and enrichment of Dengue viruses

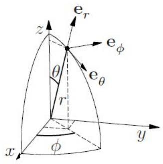
K. Lien et al., Lab Chip, 2007, 7, 868–875 | 869
DOI: 10.1039/b700516d

Unbounded Stokes flow around a sphere: The viscous drag force

Creeping flow (*Re* <<1): Acceleration can be ignored / inertial forces $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$.
The fluid is further slowed down due to viscous forces when passing the bead surface.

$$\boxed{\nabla^2 \mathbf{v} = \frac{1}{\eta}\nabla p}$$

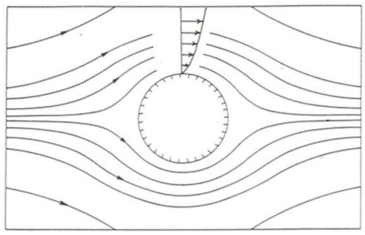
Boundary conditions
$v(a) = 0$ and $v(\infty) = v_0$



$$v_r = +V_0 \cos\theta \left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right]$$

$$v_\theta = -V_0 \sin\theta \left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]$$

⇨ Velocity field $v(r,\theta)$ in terms of a power series in a/r (spherical coordinates)



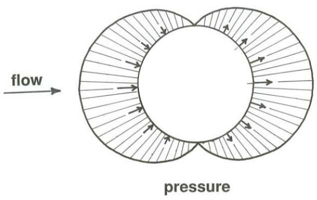
The flow pattern is symmetrical front to back.

for more details: **J. Kirby, *Micro- and nanoscale fluid mechanics***

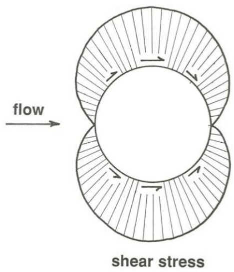
Unbounded Stokes flow around a sphere: The viscous drag force

$$p = p^* - \frac{3\eta V_0}{2a}\cos\theta \qquad \sigma'_{\theta r} = -\frac{3\eta V_0}{2a}\sin\theta \qquad (3.126)$$

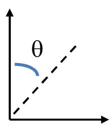
Pressure field on the sphere | *Shear stress on a sphere*



Pressure field on the sphere (radius a, p = ambient pressure)*

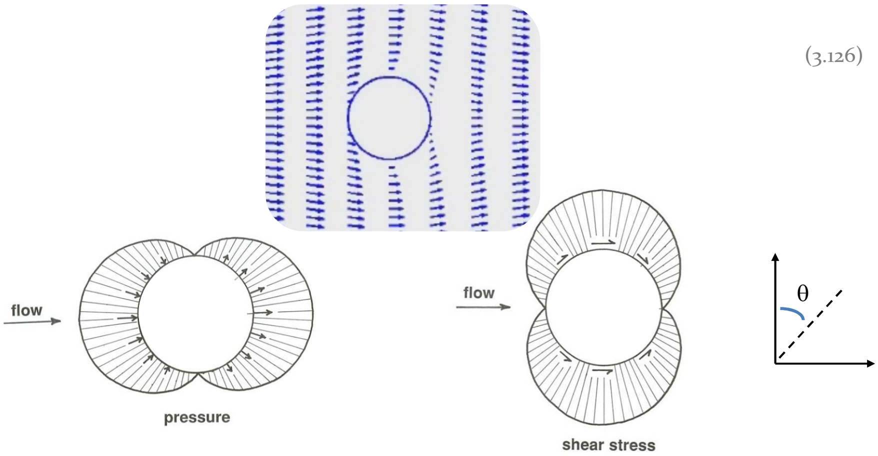


Shear stress on the surface of a sphere (radius a)



Unbounded Stokes flow around a sphere: The viscous drag force

(3.126)



Pressure field on the sphere (radius a, p = ambient pressure)*

Shear stress on the surface of a sphere (radius a)

Unbounded Stokes flow around a sphere: The viscous drag force

$$p = p^* - \frac{3\eta V_0}{2a}\cos\theta \qquad \sigma'_{\theta r} = -\frac{3\eta V_0}{2a}\sin\theta \qquad (3.126)$$

Pressure field on the sphere *Shear stress on a sphere*

⇨ The drag force $\mathbf{F}_d$ can be derived from the stress tensor $\boldsymbol{\sigma}$ as integral over the surface force densities (including the normal p components).

'Stokes Law' for the viscous drag force on a sphere

$$\boxed{F_{\text{drag}} = 6\pi\eta\, a V_0} \qquad (3.127)$$

accurate for $Re < 0.2$

Corrections:

⇨ Drag coefficient starts deviating for $\sim Re \geq 0.2$ $F_{drag} = 6\pi\eta a V_0\,(1 + 0.15 \cdot Re^{0.687})$

$0.2 < Re < 500$ -1000

⇨ Drag on a sphere will be up to a factor 3 higher in the vicinity of a solid wall.

Life at small scale and low Re numbers

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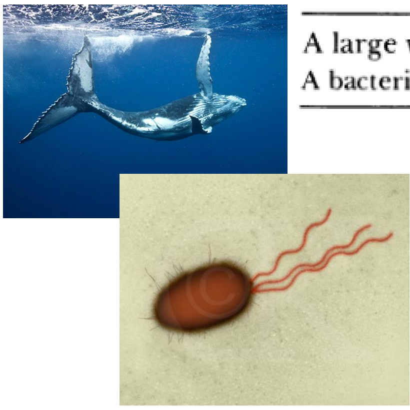
https://www.youtube.com/watch?time_continue=4&v=2kkfHj3LHeE



	Re
A large whale swimming at 10 m s^{-1}	~ 10^{8}
A bacterium, swimming at 0.01 mm s^{-1}	~ 10^{-5}

Normal conditions for human swimming are *Re* ~ 10^4 - 10^6

Can you swim like a bacteria ?

⇨ *e.g.* at ***Re* = 10^{-5}**

Propulsion mechanisms at high *Re* (*e.g.* humans, fish, etc.) are based on inertial effects, such as fins.

...but, inertia is totally irrelevant in the life of a microorganism, *i.e.* for swimming at low *Re*-number !

4/00

Life at small scale and low Re numbers

EPFL

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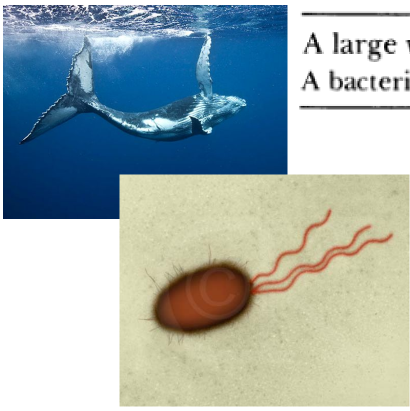

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Can you swim like a bacteria ?

⇨ *e.g.* at ***Re* = 10^{-5}**

Human swimming with v ≈ 0.1 mm/h in honey

(L = 2 m, η = 10 Pa·s, ρ = 1.5 kg/l)

May be a microorganism feels like this !

Propulsion mechanisms at high *Re* (*e.g.* humans, fish, etc.) are based on inertial effects, such as fins.

...but, inertia is totally irrelevant in the life of a microorganism, *i.e.* for swimming at low *Re*-number !

Life at small scale and low Re numbers

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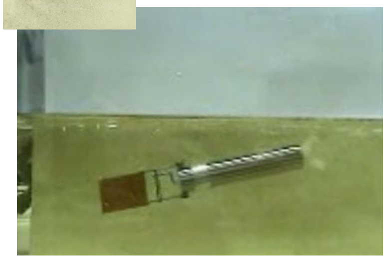
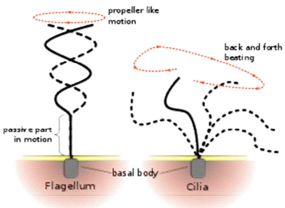


Rubber band powered toy that tries to paddle forward viscous corn syrup.

At low *Re*-numbers any reciprocal motion (even if fast in one direction and slow in the return direction) does not result in forward motion due to the reversibility of Stokes flow.

https://www.youtube.com/watch?time_continue=4&v=2kkfHj3LHeE

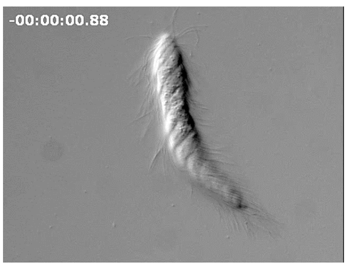
EPFL



Life in moving fluids: the physical biology of flow by S. Vogel (1996)

⇨ Microorganism have developed propulsion mechanisms, such as flagella or cilia, working as a flexible oar or as a corkscrew.

Deforming the shape of the paddle breaks the symmetry of the stroke, creating more drag on the power stroke than on the recovery stroke.



PARAMECIUM (50 to 330 µm, an abundant genus of unicellular ciliates) covered with hair-like cilia.

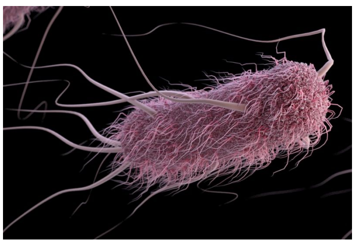


Illustration of an *Escherichia coli* based on a SEM micrograph. These bacteria use flagella for propulsion. (A. Eckert and J. Oosthuizen)

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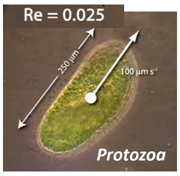
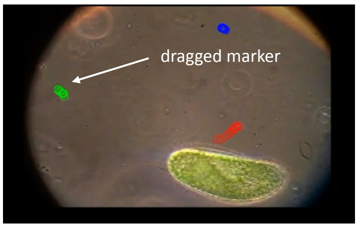
⇨ Swimming in water on microscale becomes very difficult, due to viscous drag (although the viscosity is very low, η = 1 mPa·s).

EPFL





A protozoa (length 250 µm, 100 µm/s) drags water at a distances up to 250 µm (and more).

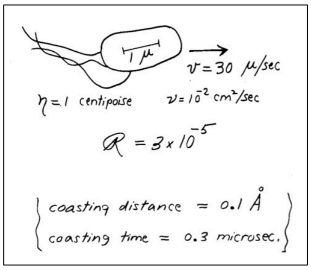
Added mass ~200 µg, *i.e.* 100 x cell mass of ~2 µg

Try to swim with 10 tons attached to our feet !

http://www.youtube.com/watch?v=gZk2bMaqs1E

A bacteria typically moves at 20-40 µm/s.

It takes him about 0.1 Å and 0.3 µs to stop.



E.M. Purcell, *Life at low Re number,* America Journal of Physics, Vol. 45, p. 3-11 (1997)

Dusenbery, David B. (2009). *Living at Micro Scale*, Harvard University Press, Cambridge.